Mertens' Conjecture disproved

by Herman te Riele *

By means of a rather simple computer program it has been established that

$$\sigma_n(t) \approx 1.0615\ ,$$

for $n = 2000$ and

$$t = t_0 = -\ 14045\ 28968\ 05929\ 98046\ 79036\ 16303\ 99781\ 12740\ 05919\ 99789\ 73803\ 99659\ 60762.52150\ 5$$

where

$$\sigma_n(t) = 2\sum_{j=1}^{n} \kappa\left[\frac{\gamma_j}{\gamma_n}\right] \frac{\cos(\gamma_j t - \pi\psi_j)}{|\rho_j \zeta'(\rho_j)|},\quad -\infty < t < \infty,$$

$$\kappa(\tau) = (1-\tau)\cos(\pi\tau) + \pi^{-1}\sin(\pi\tau),\quad 0 \leq \tau \leq 1,$$

$\rho_j = ½ + i\gamma_j,\ 1 \leq j \leq n$, is the j-th non-trivial simple zero of the Riemann zeta function,

and

$$\pi\psi_j = arg(\rho_j \zeta'(\rho_j)),\ 1 \leq j \leq n,$$

where *arg* stands for the usual argument of a complex number. Consequently, the old conjecture of F. Mertens ([3], p.779) that $|M(x)| < \sqrt{x}$ for all $x > 1$, where $M(x) = \Sigma_{n \leq x}\mu(n)$ and μ is the Möbius function, is *false*, since every value of $\sigma_n(t)$ is a lower bound for $\lim\sup_{x\to\infty} M(x)\cdot x^{-½}$ ([1], p. 329). Mertens' conjecture would have implied the Riemann hypothesis [5], p. 320).

The major problem in this joint project of CWI and Bell Laboratories was to find a value of t for which $\sigma_n(t) > 1$. The number t_0, given above, was found by Andrew Odlyzko by using the so-called *lattice basis reduction algorithm* ([2], pp. 516-525). t_0 has the remarkable property that in 70 of the 2000 terms of $\sigma_{2000}(t_0)$ *all* cosine-values are very close to $+1$. In other words, in these 70 terms *all* numbers $\gamma_j t_0 - \pi\psi_j$ are very close to a multiple of 2π. The problem of finding such a candidate t_0 is known as the problem of *inhomogeneous diophantine approximation* or *Kronecker approximation.* That it is possible to solve this problem for 70 terms has been unthinkable until very recently.

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* This announcement reports on joint research with Andrew Odlyzko of Bell Laboratories (Murray Hill, New Jersey, USA)

It took the CRAY 1 computer of Bell Labs about three hours and a considerable amount of memory to find t_0.

A second problem was the high-precision computation of the imaginary parts γ_j of the first 2000 non-trivial zeros of the Riemann zeta function which was essential for the computation of

$$\cos(\gamma_j t_0 - \pi\psi_j), \quad 1 \leqslant j \leqslant n$$

in the above formula for $\sigma_n(t)$. This was carried out by the author on the CDC CYBER 175 - 750 computers of SARA. The zeros were computed with an accuracy of 105 decimal digits, with the help of a special multi-precision package of R.P. Brent of the Australian National University. The zeros were computed with the well-known Newton process from 28 digit approximations already known ([4]). The total amount of (nightly) computer time needed was about 40 hours.

The communication between CWI and Bell Labs was not maintained by letters, but by electronic mail transmitted via the VAX-computers of the two institutes. This enabled the participants to exchange their data and results with a very high speed, frequency and reliability. Without this facility the whole project would have taken at least six months. Now it was completed in less than two months.

Only five years ago the author still believed that Mertens' conjecture could not be disproved "using present day computers and current techniques" ([4], p. 356). Now, the disproof shows how rapidly new algorithms and super-fast computers have been developed in the past few years.

References

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(received, October 21, 1983).